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On constructible graphs, infinite bridged graphs and weakly cop-win graphs

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Abstract

A (finite or infinite) graph G is *constructible* if there exists a well-ordering \leq of its vertices such that, for every vertex x which is not the smallest element, there is a vertex $y < x$ which is adjacent to x and to every neighbor z of x with $z < x$. We prove that every Helly graph and every connected bridged graph are constructible. From the latter result we deduce new characterizations of bridged graphs, and also that any connected bridged graph is ‘moorable’, a property which implies various fixed-point properties (see Chastand, *Classes de graphes compacts faiblement modulaires*, These de doctorat, Université Claude Bernard, Lyon 1, 1997.), and thus that any connected bridged graph is a retract of the Cartesian product of its blocks. We also solve a problem of Hahn et al. (personal communication) by proving that any finite subgraph of a bridged (resp. constructible) graph G is contained in a finite induced subgraph K of G which is bridged (resp. constructible). Moreover, the vertex set of K is a geodesically convex subset of $V(G)$ whenever G is locally finite or contains no infinite paths. Finally, we study some relations between constructible graphs and a weakening of the concept of cop-win graphs. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Roughly, a graph G is said to be *dismantlable* if its vertices can be removed one after the other in such a way that a vertex x can be taken off the currently remaining subgraph G_x of G if there exists a vertex y in G_x which is adjacent to x and to all neighbors of x in G_x . On the other hand, we will say that a graph G is *constructible* if it can be built vertex after vertex so that a vertex x can be added to the currently

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constructed induced subgraph G_x of G if there exists a vertex y of G_x which is adjacent in G to x and to all neighbors of x belonging to G_x .

These opposite concepts, which are clearly equivalent for finite graphs, are quite different for infinite graphs. For example, as we will see, double rays are constructible but not dismantlable. The concept of dismantlability, and the restricted one of strong dismantlability in the case of infinite graphs, has been mainly used to prove invariant simplex properties and convexity properties such as Helly-type theorems, in particular for Helly graphs and bridged graphs (see for instance Polat [13,14,16]).

There is a close relationship between these two concepts and the one of cop-win graphs which was introduced by Nowakowski and Winkler [11] which we will now recall. Consider the following game, which is played on a given graph G . There are two players, the *cop* and the *robber*. They move alternatively, starting with the cop. Each player's first move consists of choosing a vertex at which to start. At each subsequent move a player may choose either to remain where he is or to move to an adjacent vertex. The object of the game for the cop is to catch the robber, that is, occupy the same vertex as him, and for the robber, to prevent this from happening. The graphs on which the cop can always win are called *cop-win* by Nowakowski and Winkler [11] who characterized them. In particular they showed that a *finite graph is cop-win if and only if it is dismantlable*. This result was independently proved by Quillot [18].

As we already said, the concept of dismantlability and the one of constructibility, introduced in this paper, coincide for finite graphs. Moreover, the latter seems to be a better generalization to the infinite case since all Helly graphs and all connected bridged graphs are constructible. In fact, for these graphs, breadth-first search always gives an ordering of vertices that can be induced by constructibility. Also, two characterizations of bridged graphs given by Anstee and Farber [1] are easily generalized by using constructibility. Furthermore, this concept of constructibility is a good tool for studying different problems about some infinite graphs. For instance, it enables us to solve a problem of Hahn et al. [9] and one of Chastand [3], and to study an extension of the class of infinite cop-win graphs.

A class \mathcal{C} of graphs is said to be *compact closed* if, whenever a graph G is such that each of its finite subgraphs is contained in a finite-induced subgraph of G which belongs to the class \mathcal{C} , then the graph G itself belongs to \mathcal{C} . In this paper we will have to deal with the dual concept. We will say that a class \mathcal{C} of graphs is *dually compact closed* if, for every infinite $G \in \mathcal{C}$, each finite subgraph of G is contained in a finite induced subgraph of G which belongs to \mathcal{C} . The class of all chordal graphs is clearly dually compact closed because every induced subgraph of a chordal graph is chordal. Also, due to results of Pesch [12] and of Jawhari et al. [10, Theorem IV-1.2.2], the class of absolute retracts of reflexive graphs, alias Helly graphs, is dually compact closed. Hahn et al. [9] suggested to study the dually compact closed classes of graphs, and in particular to determine if the class of bridged graphs is dually compact closed. In [8] Hahn et al. gave a partial answer to this problem by proving that every finite subgraph

of a bridged graph G of diameter two is contained in a finite induced subgraph of G which is bridged and has diameter two. In Sections 3 and 5 of this paper we prove that the class of constructible graphs and the one of bridged graphs, respectively, are dually compact closed, which gives an affirmative answer to Hahn, Sauer and Woodrow's problem about bridged graphs. We also study different refinements of this result.

In his thesis [3] Chastand introduced the class of pre-median graphs, a subclass of the class of weakly modular graphs. Particular instances of pre-median graphs are median graphs, quasi-median graphs, pseudo-median graphs and bridged graphs. Among several properties that are shared by these graphs, some are only enjoyed by a few of them which are said to be *moorable*. The concept of moorable graphs was first introduced by Tardif [20] for median graphs. Chastand showed that pseudo-median graphs and quasi-median graphs are moorable. In Section 4 we prove that bridged graphs are also moorable, and we give one consequence of this property.

In Section 6 we weaken the concept of cop-win graphs by deciding that the cop wins, not only if he catches the robber, but if he can force him to run straight ahead by never moving to a vertex more than once. Of course in the finite case the cop-win concept and the weakly one coincide. We then show that these so-called *weakly cop-win graphs* are, as in the finite case, closely related to constructible graphs, and that, in particular, both Helly graphs and bridged graphs are weakly cop-win.

2. Notation

The graphs we consider are undirected, without loops and multiple edges. A complete graph will be simply called a *simplex*. If $x \in V(G)$, the set $N_G(x) := \{y \in V(G) : \{x, y\} \in E(G)\}$ is the *neighborhood* of x in G . For $A \subseteq V(G)$ we denote by $G[A]$ the subgraph of G induced by A , and we set $G - A := G[V(G) - A]$.

A *path* $P = \langle x_0, \dots, x_n \rangle$ is a graph with $V(P) = \{x_0, \dots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(P) = \{\{x_i, x_{i+1}\} : 0 \leq i < n\}$. A *ray* or *one-way infinite path* $\langle x_0, x_1, \dots \rangle$ and a *double ray* or *two-way infinite path* $\langle \dots, x_{-1}, x_0, x_1, \dots \rangle$ are defined similarly. A graph is *rayless* if it contains no ray. A path $P = \langle x_0, \dots, x_n \rangle$ is called an (x_0, x_n) -*path*, x_0 and x_n are its *endpoints*, while the other vertices are called its *internal* vertices, $n = |E(P)|$ is the *length* of P .

The usual *distance* in a connected graph G between two vertices x and y , that is the length of an (x, y) -*geodesic* (i.e., shortest (x, y) -path) in G , is denoted by $d_G(x, y)$. A subgraph H of G is *isometric* in G if $d_H(x, y) = d_G(x, y)$ for all vertices x and y of H . If x is a vertex of G and r a non-negative integer, the set $B_G(x, r) := \{y \in V(G) : d_G(x, y) \leq r\}$ is the *ball of center x and radius r in G* , and the set $S_G(x, r) := \{y \in V(G) : d_G(x, y) = r\}$ is the *sphere of center x and radius r in G* . The smallest integer r such that $V(G) \subseteq B_G(x, r)$ for some vertex x is the *radius* of G .

3. Constructible graphs

If x and y are two vertices of a graph G , then we say that x is *dominated* by y in G if $B_G(x, 1) \subseteq B_G(y, 1)$. We will first recall the definition of a dismantlable graph.

Definition 3.1. A graph G is said to be *dismantlable* if there is a well-order \preceq on $V(G)$ such that, every vertex x which is not the greatest element of $(V(G), \preceq)$, if such a greatest element exists, is dominated by some vertex $y \neq x$ in the subgraph of G induced by the set $\{z \in V(G) : x \preceq z\}$. The well-order \preceq on $V(G)$, and the enumeration of the vertices of G induced by \preceq , will be called a *dismantling order* and a *dismantling enumeration*, respectively.

Definition 3.2. A graph G is said to be *constructible* if there is a well-order \leq on $V(G)$ such that, every vertex x which is not the smallest element of $(V(G), \leq)$ is dominated by some vertex $y \neq x$ in the subgraph of G induced by the set $\{z \in V(G) : z \leq x\}$. The well-order \leq on $V(G)$, and the enumeration of the vertices of G induced by \leq , will be called a *constructing order* and a *constructing enumeration*, respectively.

Remark 3.3. (1) Clearly a finite graph G is dismantlable if and only if it is constructible. In fact, in this case, a constructing order on $V(G)$ is the dual of a dismantling order on this set. This may not be true if G is infinite. There are constructible graphs which are not dismantlable, and dismantlable ones which are not constructible, as is shown by the two following examples.

Example 1. A double ray $D = \langle \dots, -1, 0, 1, \dots \rangle$ is not dismantlable since no vertex of D is dominated, but it is constructible : $0, 1, -1, 2, -2, \dots$ is a constructing enumeration of $V(D)$.

Example 2. Let $\langle a_0, a_1, \dots \rangle$ and $\langle b, c, d \rangle$ be two disjoint paths, and let G be the graph obtained by joining the vertices b and d to a_n for every non-negative integer n . This graph G is dismantlable since a_0, a_1, \dots, b, c, d is a dismantling order on $V(G)$. It is not constructible because if \leq was a constructing order on $V(G)$, if n was such that $a_n < a_p$ for every $p \neq n$, and if x was the greatest vertex of the cycle $\langle a_n, b, c, d, x_n \rangle$ with respect to \leq , then x would not be dominated in $G[\{y \in V(G) : y \leq x\}]$, contrary to the definition of a constructing order.

(2) A constructing order may be a dismantling order. Take a ray $R = \langle 0, 1, \dots \rangle$. Then $0, 1, 2, \dots$ is both a constructing enumeration and a dismantling enumeration.

(3) Let \leq be a constructing order on the vertex set of a graph G with u as the smallest element. Any self-map Δ of $V(G)$ such that $\Delta(u) = u$ and, for each vertex $x \in V(G - u)$, $\Delta(x)$ is a vertex of G which dominates x in $G[\{y \in V(G) : y \leq x\}]$, will

be called a *domination map* associated with \leq . The set of domination maps associated with \leq is clearly ordered by the relation $\Delta_1 \leq \Delta_2$ if and only if $\Delta_1(x) \leq \Delta_2(x)$ for every $x \in V(G)$. Since \leq is a well-order, there always exists a smallest domination map which is associated with \leq . Furthermore, because a well-order contains no infinite descending chain, for every domination map Δ and every $x \in V(G)$, there exists a non-negative integer n such that $\Delta^n(x) = u$.

We recall that if G and H are two graphs, a map $f: V(G) \rightarrow V(H)$ is a *contraction* if f preserves or contracts the edges, i.e., if $f(x) = f(y)$ or $\{f(x), f(y)\} \in E(H)$ whenever $\{x, y\} \in E(G)$. A contraction f from G onto an induced subgraph H of G is a *retraction*, and H is a *retract* of G , if the restriction of f to $V(H)$ is the identity. Note that this is slightly different from the usual definition of a retract.

Proposition 3.4. *Let \leq be a constructing order on the vertex set of a graph G . Then, for every $x \in V(G)$, the induced subgraph $G[\{y \in V(G): y \leq x\}]$ is a retract of G .*

Proof. Let u be the smallest element of $(V(G), \leq)$, Δ a domination map which is associated with \leq , and let $x \in V(G)$. For every $y \in V(G)$, put $G_y := G[\{z \in V(G): z \leq y\}]$, and denote by $n(y)$ the smallest non-negative integer n such that $\Delta^n(y) \in V(G_x)$. The existence of $n(y)$ follows from the fact that, by Remark 3.3(3), $\Delta^n(y) = u$ for some non-negative integer n , and $u \in V(G_x)$. Put $\Psi(y) := \Delta^{n(y)}(y)$. We will show that the map Ψ is a retraction from G onto G_x .

Clearly, $\Psi(y) = y$ for every $y \in V(G_x)$. Let y_0 and y_1 be two adjacent vertices of G . For $i = 0, 1$ consider the strictly decreasing sequence $y_i = \Delta^0(y_i) > \dots > \Delta^{n(y_i)}(y_i)$. If $\Delta^{n_0}(y_0) = \Delta^{n_1}(y_1)$ for some $n_i \leq n(y_i)$, $i = 0, 1$, then $\Psi(y_0) = \Psi(y_1)$. Suppose that $\Delta^{n_0}(y_0) \neq \Delta^{n_1}(y_1)$ for every $n_i \leq n(y_i)$, $i = 0, 1$. Note that, if $\Delta^{n_0}(y_0)$ and $\Delta^{n_1}(y_1)$ are adjacent, then, by the definition of a constructing order and that of the relation of domination, either $\Delta^{n_0+1}(y_0)$ and $\Delta^{n_1}(y_1)$ are adjacent or $\Delta^{n_0}(y_0)$ and $\Delta^{n_1+1}(y_1)$ are adjacent according to whether $\Delta^{n_0}(y_0) \leq \Delta^{n_1}(y_1)$ or $\Delta^{n_1}(y_1) \leq \Delta^{n_0}(y_0)$. Therefore, since y_0 and y_1 are adjacent, we can easily prove by induction, that $\Psi(y_0)$ and $\Psi(y_1)$ are also adjacent. Thus $\Psi(y_0)$ and $\Psi(y_1)$ either coincide or are adjacent, which proves that Ψ is a contraction, hence a retraction from G onto G_x . \square

From this result we deduce immediately:

Corollary 3.5. *Let \leq be a constructing order on the vertex set of a graph G . Then \leq is distance-preserving, that is, for every $x \in V(G)$, the induced subgraph $G[\{y \in V(G): y \leq x\}]$ is an isometric subgraph of G .*

We will now prove our first result related to Hahn et al.'s problem [9].

Theorem 3.6. *The class of all constructible graphs is dually compact closed.*

Proof. Let G be a constructible graph, and let A be a finite set of vertices of G . Let \leq be a constructing order on $V(G)$ with some vertex u as the smallest element, and let Δ be a domination map associated with \leq . By Remark 3.3(3), for every $a \in A$, there exists a non-negative integer $n(a)$ such that $\Delta^{n(a)}(a) = u$. Let $H := G[\bigcup_{a \in A} \{\Delta^i(a) : 0 \leq i \leq n(a)\}]$. This graph H is finite and contains $G[A]$. Furthermore, the restriction of \leq to $V(H)$ is obviously a constructing order on $V(H)$, which proves the result. \square

We will see that, for different classes of graphs, a useful tool for obtaining constructing orders is the concept of breadth-first search (BFS). We recall that a BFS of a given graph G with n vertices produces an enumeration x_1, \dots, x_n of the vertices of G in the following way. We number with 1 some vertex of G and put it at the head of an empty queue. At the i th step we number and add at the end of the current queue, all still unnumbered neighbors of the head x_i of the queue, then we remove x_i .

Definition 3.7. Let G be a connected graph. A well-order \leq on $V(G)$ is called a *BFS-order* if there exists a family $(A_x)_{x \in V(G)}$ of subsets of $V(G)$ such that, for every $x \in V(G)$:

- (i) $x \in A_x$;
- (ii) if $x \leq y$, then A_x is an initial segment of A_y with respect to the induced order;
- (iii) $A_x = A_{(x)} \cup N_G(x)$ where $A_{(x)} := \{x\}$ if x is the least element of $(V(G), \leq)$, and otherwise $A_{(x)} := \bigcup_{y < x} A_y$.

The vertex x will be called the *father* of each element of $A_x - A_{(x)}$. We will denote by ϕ , and call *father function*, the self-map of $V(G)$ such that $\phi(x)$ is the father of x , for every $x \in V(G)$.

Note that, by (i) and (ii), $x \in A_{(x)}$ for every vertex x of G . Besides, if G is finite, then the queue whose head is x in the BFS is the linearly ordered set $(\{y \in A_{(x)} : x \leq y\}, \leq)$. Also notice that if u is the smallest element of $(V(G), \leq)$, then clearly, for all vertices x and y of G , $x \leq y$ implies $d_G(u, x) \leq d_G(u, y)$, and $d_G(u, x) < d_G(u, y)$ implies $x < y$. In particular $d_G(u, x) = d_G(u, \phi(x)) + 1$.

Lemma 3.8 (Polat [16, Lemma 3.6]). *There exists a BFS-order on the vertex set of any connected graph.*

Note that a BFS-order is not necessarily a constructing order, and a constructing order is not necessarily a BFS-order. For example, consider the graph G which is formed by a ray $\langle 0, 1, 2, \dots \rangle$ and two other vertices a and b which are adjacent to all vertices of the ray. Then $0, a, b, 1, 2, 3, \dots$ is both a BFS-order and a constructing order; $a, 0, 1, 2, 3, \dots, b$ is a BFS-order but not a constructing order since b is not dominated by any vertex of G ; $a, 0, b, 1, 2, \dots$ is a constructing order but not a BFS-order since b is not a neighbor of a .

Furthermore, there exist constructible graphs such that none of their constructing orders is a BFS-order, as is shown by the following example. Let $X = \{x_n: n \in \mathbb{N}\}$, $Y = \{y_n: n \in \mathbb{N}\}$ and $\{z\}$ be three disjoint sets of vertices. Let G be the graph such that $V(G) := X \cup Y \cup \{z\}$ and $E(G) := \bigcup_{n \in \mathbb{N}} (\{x_n, x_{n+1}\} \cup \{z, x_n\} \cup \{\{x_n, y_p\}: p \leq n \leq p+3\})$. This graph is dismantlable: $x_0, y_0, x_1, y_1, \dots, x_n, y_n, \dots, z$ is a dismantling enumeration. It is also constructible. For instance, $z, x_0, y_0, x_1, y_1, \dots, x_n, y_n, \dots$ is a constructing enumeration. But in a BFS-order \leq the vertex z will be in the first 10 places since it is adjacent to all vertices x_n 's, and for this reason these vertices x_n 's will be placed before infinitely many vertices y_n 's. Hence, since each vertex y_n is adjacent to four vertices x_n 's, none of these vertices y_n 's will be dominated in the subgraph $\{u: u \leq y_n\}$.

On the contrary, for some classes of constructible graphs, any BFS-order is a constructing order. This is the case, as we will see, of Helly graphs and of bridged graphs.

We recall that a *Helly graph* is a connected graph for which any (finite or infinite) family of pairwise non-disjoint balls has a nonempty intersection. Helly graphs are also known as absolute retracts of reflexive graphs (see [10]).

Theorem 3.9. *Every Helly graph G is constructible. Moreover any BFS-order on $V(G)$ is a constructing order.*

Proof. Let G be a Helly graph, and let \leq be a BFS-order on $V(G)$ with u as the smallest element. Let $x \in V(G - u)$, $G_x := G[\{y \in V(G): y \leq x\}]$ and $r := d_G(u, x)$. We have to prove that x is dominated by some vertex $y \neq x$ in G_x .

This is obvious if $r=1$. Suppose $r > 1$. The balls $B_G(u, r-1)$, $B_G(x, 1)$ and $B_G(y, 1)$ for every $y \in N_{G_x}(x)$ are pairwise non-disjoint, in particular because $d_G(u, y) \leq r$ for every $y \in N_{G_x}(x)$. Hence, since G is a Helly graph, their intersection is nonempty. Let z be an element of their intersection. Then z is a vertex of G_x whose distance to u is $r-1$ and which, by definition, dominates x in G_x . Therefore \leq is a constructing order on $V(G)$. \square

In the next few sections we shall have to deal, for some graph G , with a constructing order admitting a domination map which is a self-contraction of G . Therefore we will complete this section with general sufficient conditions for a constructing order to have such a domination map.

Lemma 3.10. *Let \leq be a constructing order on the vertex set of a graph G admitting an associated domination map Δ such that, for every edge $\{x, y\}$ of G , $x < y$ implies $\Delta(y) \leq x$. Then Δ is a self-contraction of G .*

Proof. Let $\{x, y\} \in E(G)$ with $x < y$. Then $\Delta(y) \leq x$. Hence $\Delta(x)$ and $\Delta(y)$ either coincide or are adjacent since $\Delta(x)$ dominates x in the subgraph of G induced by $\{z \in V(G): z \leq x\}$. \square

The condition given in the preceding lemma is not necessary. For example, consider the complete graph K_4 with $V(K_4) = \{1, 2, 3, 4\}$. The natural order \leq on the set of

integers is a constructing order on K_4 . Define Δ by $\Delta(1) = \Delta(2) = 1$, $\Delta(3) = 2$ and $\Delta(4) = 3$. Then Δ is a domination map associated with \leq , and is also a self-contraction of G since G is a complete graph, but $\Delta(4) > 2$ though $2 < 4$.

Corollary 3.11. *Let \leq be a BFS-order on the vertex set of a graph G whose father function ϕ is a domination map associated with \leq . Then ϕ is a self-contraction of G .*

Proof. Denote by u the smallest vertex of G with respect to \leq . Let $\{x, y\} \in E(G)$. W.l.o.g. we can suppose that $x < y$. Then $\phi(y)$ and x either coincide or are adjacent. We are done if $\phi(y) = x$ since $\phi(x)$ equals x or is adjacent to x according to whether $x = u$ or $x \neq u$. If $\phi(y) \neq x$, then $\phi(y) < x$ by the definition of the father of x since x and y are adjacent. Whence the result by Lemma 3.10. \square

4. Bridged graphs

We recall that a graph is *bridged* if it contains no isometric cycles of length greater than three. We will give several characterizations of bridged graphs.

The *interval* $I_G(x, y)$ of two vertices x and y of a graph G is the set of vertices of all (x, y) -geodesics in G . A set A of vertices of a graph G is *geodesically convex*, for short *convex*, if it contains the interval $I_G(x, y)$ for all $x, y \in A$. We will also say that a subgraph of G is *convex* if its vertex set is a convex subset of $V(G)$. We recall that, by Soltan and Chepoi [19] and by Farber and Jamison [7], *the balls of a bridged graph are convex*.

To prove the next theorem we will recall a result of Chepoi [5] which was proved for finite graphs only, but without use of finiteness.

Lemma 4.1. *Let \leq be a BSF-order on the vertex set of a bridged graph G , u its smallest element and ϕ its father function. Let x and y be two adjacent vertices of G which are equidistant to u . Then $\phi(x)$ and $\phi(y)$ either coincide or are adjacent. In addition, if $y < x$, then y is adjacent to $\phi(x)$.*

Lemma 4.2 (Anstee and Farber [1, Corollary 2.6]). *A finite connected graph G is bridged if and only if G is dismantlable and has no induced cycles of length 4 or 5.*

Note that in the original statement of this result, Anstee and Farber used the term of cop-win instead of dismantlable in view of the results of Nowakowski and Winkler [11].

Theorem 4.3. *Let G be a connected graph. The following assertions are equivalent:*

- (i) G is bridged.
- (ii) Any BFS-order on $V(G)$ is a constructing order for which the father function is an associated domination map.
- (iii) G is constructible and has no induced cycles of length 4 or 5.
- (iv) Every isometric subgraph of G (including G itself) is constructible.

Proof. (i) \Rightarrow (ii): Let G be a connected bridged graph, and \leq be a BFS-order on $V(G)$ with u as the smallest element. Let $x \in V(G - u)$, $G_x := G[\{y \in V(G): y \leq x\}]$ and $r := d_G(u, x)$. We will prove that x is dominated by its father $\phi(x)$ in G_x .

This is obvious if $r = 1$. Suppose $r > 1$, and let $y \neq \phi(x)$ be a neighbor of x in G_x . We have to show that y is adjacent to $\phi(x)$.

If $d_G(u, y) = r$, then we are done by Lemma 4.1. Assume that $d_G(u, y) = r - 1 = d_G(u, \phi(x))$. Since $d_G(y, \phi(x)) \leq 2$, the convexity of the ball $B_G(u, r - 1)$ implies that y and $\phi(x)$ are adjacent, otherwise x would belong to this ball, contrary to the definition of r .

(ii) \Rightarrow (iii): Suppose that G satisfies (ii). Then it is constructible. Assume that it contains an induced cycle $C = \langle x_0, \dots, x_{n-1}, x_0 \rangle$ of length $n = 4$ or $n = 5$. Therefore, we can construct a BFS-order \leq on $V(G)$ satisfying the following conditions:

- x_0 is the smallest element of $V(G)$;
- x_1 is the successor of x_0 , and x_{n-1} the one of x_1 .

Since C is chordless this implies that we have:

- for $n = 4$: $x_1 = \phi(x_2)$ with $x_3 < x_2$ and x_1 and x_3 non-adjacent;
- for $n = 5$: $x_4 = \phi(x_3)$ with $x_2 < x_3$ and x_2 and x_4 non-adjacent.

Consequently in both cases ϕ is not a domination map, contrary to (ii).

(iii) \Rightarrow (i): Suppose that G satisfies (iii), and let C be a cycle of G of length greater than three. By (iii) G is constructible. Therefore, by Theorem 3.6, C is contained in a finite induced subgraph H of G which is constructible, hence dismantlable by Remark 3.3(1). By (iii), G , hence H , contains no cycles of length 4 and 5. Therefore, H is bridged by Lemma 4.2, which implies that C is not an isometric cycle of H , thus a fortiori of G .

(i) \Rightarrow (iv): This is obvious by the equivalence of (i) and (iii) and the fact that any isometric subgraph of a bridged graph is also bridged.

(iv) \Rightarrow (iii): If G satisfies (iv), then it is constructible. Furthermore, an isometric cycle of G of length 4 or 5 is induced, and thus cannot be constructible. \square

Note that the equivalences (i) \Leftrightarrow (iii) and (i) \Leftrightarrow (iv) extend to infinite graphs the results of Anstee and Farber [1, Corollary 2.6] and [1, Corollary 2.7], respectively. Furthermore the implication (i) \Rightarrow (ii) extends to infinite bridged graphs the result of Chepoi [5]. Moreover, if \leq is a BFS-order on the vertex set of a connected bridged graph G , then the preceding proof and the definition of the father $\phi(x)$ of a vertex x show that ϕ is the smallest domination map associated with \leq . Finally, we want to

mention that recently Chepoi [6] proved a result which has a close relationship with Theorem 4.3.

We recall that bridged graphs are particular instances of *pre-median graphs* (see [3]). Among pre-median graphs, those which are said to be *moorable* have interesting properties, in particular, a retraction property and invariant subgraph properties. We will prove that any bridged graph is moorable; but instead of stating the general definitions and results of pre-median graphs that we need, we will only give their corresponding counterparts for bridged graphs, which are much simpler.

A self-contraction f of a graph G is said to be a *mooring* onto a vertex u of G if $f(u) = u$ and $\{x, f(x)\}$ is an edge of $G[I_G(x, u)]$ for every vertex $x \neq u$. This concept was introduced by Tardif [20]. By [3, Proposition 9.5.2] a bridged graph G is *moorable* if, for every block (maximal 2-connected subgraph) H of G and every vertex u of H , there exists a mooring of H onto u .

Proposition 4.4. *Every bridged graph is moorable.*

We will have the stronger result that, for every connected bridged graph, there exists a mooring onto any of its vertices. From Corollary 3.11 and Theorem 4.3 we obtain the following corollary.

Corollary 4.5. *Let G be a connected bridged graph, and \leq a BFS-order on $V(G)$. Then its father function ϕ is a self-contraction of G .*

The father function ϕ is clearly a mooring of G onto u , which implies Proposition 4.4. As a consequence of this proposition we will only state the counterpart for bridged graphs of [3, Theorem 9.4.2].

Theorem 4.6. *Every connected bridged graph is a retract of the Cartesian product of its blocks.*

5. Finite subgraphs of infinite bridged graphs

We will now consider Hahn et al.'s problem about bridged graphs.

Theorem 5.1. *The class of bridged graphs is dually compact closed.*

Proof. Let H be a finite subgraph of a bridged graph G . W.l.o.g. we can suppose that G is connected. Hence, by Theorem 4.3, G is constructible. Therefore, by Theorem 3.6, H is contained in a finite induced subgraph K of G which is constructible, hence dismantlable by Remark 3.3(1). Since G is bridged and K is an induced subgraph of G , K contains no induced cycles of length 4 or 5. Therefore, K is bridged by Lemma 4.2. \square

We can give a more detailed version of the above result as follows.

Proposition 5.2. *If H is a finite subgraph of a bridged graph G such that $V(H)$ is contained in some ball $B_G(u, r)$ of G , then it is contained in a finite induced subgraph K of G which is bridged and such that $V(K) \subseteq B_G(u, r)$. In particular K can be chosen so that its radius is at most that of H .*

Proof. Suppose that $V(H) \subseteq B_G(u, r)$ for some vertex u of G and some positive integer r that we can choose to be the radius of H . Due to the convexity of each ball of G , the subgraph G' of G induced by $B_G(u, r)$ is bridged. Take a BFS-order \leq of G' beginning by u . By Theorem 4.3, \leq is a constructing order. Hence, the finite induced subgraph K of G' containing H , that is obtained by the proof of Theorem 3.6, has the required properties. \square

We recall that Hahn et al. [8] proved that *every finite subgraph H of a bridged graph G of diameter two is contained in a finite induced subgraph K of G which is bridged and has diameter two*. Note that, in this result, the fact that the finite bridged subgraph K has diameter two implies that K is an isometric subgraph of G . This brings us to the question of whether K can always be chosen to be isometric. Clearly an isometric subgraph of a bridged graph is bridged, hence this problem is a natural enhancement of the one of Hahn, Sauer and Woodrow. However this seems to be very difficult. From this point of view, we will now show some refinements of Theorem 5.1, first by giving conditions for the subgraph K to be a convex (hence a fortiori isometric) subgraph of G . Our result is directly related to the convex hulls of finite sets of vertices in bridged graphs. We will say that a graph is *interval-finite* if all its intervals are finite. First, we recall two results.

Lemma 5.3 (Chastand and Polat [4, Lemma 4.13]). *The convex hull of a finite set of vertices of an interval-finite rayless graph is finite.*

Lemma 5.4 (Polat [16, Lemma 4.5]). *Any bridged graph containing no infinite simplices is interval-finite.*

Theorem 5.5. *Let G be a bridged graph. Then every finite subgraph of G is contained in a finite convex (and hence bridged) subgraph of G if every block of G is rayless or locally finite.*

Proof. (a) First, we will assume that G itself is rayless or locally finite. W.l.o.g. we can suppose that G is connected. Let H be a finite subgraph of G .

If G is rayless, then, by Lemmas 5.3 and 5.4 the convex hull of $V(H)$ is finite. Hence, we are done. Suppose now that G is locally finite. Then, every ball of G is finite and convex because G is locally finite and bridged, respectively. Hence, the convex hull of $V(H)$ is finite, and once again we are done.

(b) Now assume that each block of G is rayless or locally finite. Let H be a finite subgraph of G . W.l.o.g. we can assume that H is connected. Since it is finite, H only meets finitely many blocks of G . Furthermore, since H is connected, the convex hull of $V(H)$ is the union of the convex hulls of the intersections of $V(H)$ with the vertex set of each block of G . Moreover, since any block of a bridged graph is clearly a convex subgraph of this graph, and is itself a bridged graph, and because each block of G is rayless or locally finite, the convex hull of the intersection of $V(H)$ with any block of G is finite by (a). Therefore, the convex hull of $V(H)$ is finite. \square

We will now generalize the result of Hahn et al. [8] by considering bridged graphs of radius 2.

Theorem 5.6. *Let G be a bridged graph of radius 2 such that $G[N_G(u)]$ contains no infinite simplices for some vertex u of G with $V(G) = B_G(u, 2)$. Then every finite subgraph H of G is contained in a finite isometric (and hence bridged) subgraph of G .*

We need several lemmas. In the following we will suppose that G is a bridged graph such that $V(G) = B_G(u, 2)$ for some vertex u . First we recall that a cycle C of a graph G is *well-bridged* if, for every $x \in V(C)$, either the neighbors of x in C are adjacent, or $d_G(x, y) < d_C(x, y)$ for some antipode y of x in C (an *antipode* of x in C is a vertex of C at maximum distance from x in C).

Lemma 5.7 (Farber and Jamison [7, Theorem 3.1]). *Every cycle of a bridged graph is well-bridged.*

Lemma 5.8. *For every $x \in S_G(u, 2)$, $N_G(x) \cap S_G(u, 1)$ induces a simplex in G .*

Proof. Let $a, b \in N_G(x) \cap S_G(u, 1)$. Since $d_G(u, x) = 2$, a and b must be adjacent, otherwise, the cycle $\langle u, a, x, b, u \rangle$ would be isometric. \square

Lemma 5.9. *If x and y are two adjacent vertices in $S_G(u, 2)$, then there exists a vertex $z \in S_G(u, 1)$ which is adjacent to both x and y .*

Proof. Bridged graphs are particular instances of weakly modular graphs (see Bandelt and Chepoi [2]). These graphs are characterized by two properties: the triangle property and the quadrangle property; and Lemma 5.9 is an immediate consequence of the triangle property. \square

If the distance of two vertices in $S_G(u, 2)$ is 4, then there is clearly a geodesic joining these two vertices whose internal vertices belong to $B_G(u, 1)$. As we will see this property also holds if the distance between these two vertices is 3.

Lemma 5.10. *Let $x, y \in S_G(u, 2)$ be such that $d_G(x, y) = 3$. Then there exists an (x, y) -geodesic in G whose internal vertices belong to $S_G(u, 1)$.*

Proof. Let $\langle x, a, b, y \rangle$ be an (x, y) -geodesic in G . We will assume that at least a or b belongs to $S_G(u, 2)$, otherwise we are done. Hence, we can distinguish two cases.

Case 1: $a \in S_G(u, 2)$ and $b \in S_G(u, 1)$.

Let $\langle u, x', x \rangle$ be a (u, x) -path. Since the cycle $C = \langle u, x', x, a, b, u \rangle$ is well-bridged and since x and a are the antipodes of u in C with $d_G(u, x) = d_C(u, a) = 2$, the vertices x' and b must be adjacent. Hence $\langle x, x', b, y \rangle$ is an (x, y) -geodesic, and we are done.

Case 2: $a, b \in S_G(u, 2)$.

By Lemma 5.9, there is a vertex x' (resp. y') in $S_G(u, 1)$ which is adjacent to both x and a (resp. y and b). These vertices x' and y' are distinct, otherwise $d_G(x, y)$ would be 2, contrary to the hypothesis. Since the cycle $C = \langle u, x', a, b, y', u \rangle$ is well-bridged and since a and b are the antipodes of u in C with $d_G(u, a) = d_C(u, b) = 2$, the vertices x' and y' must be adjacent. Therefore $\langle x, x', y', y \rangle$ is an (x, y) -geodesic, and we are done. \square

We will now consider the case of the pairs of vertices in $S_G(u, 2)$ whose distance is 2.

Lemma 5.11. *Let x_0 and x_1 be two vertices in $S_G(u, 2)$ whose distance in G is 2, but such that no vertex in $S_G(u, 1)$ is adjacent to both x_0 and x_1 . Then, for every vertex $y \in S_G(u, 2)$ which is adjacent to both x_0 and x_1 , $N_G(x_i) \cap S_G(u, 1) \subseteq N_G(y)$ for $i = 0, 1$.*

Proof. Let $y \in S_G(u, 2)$ which is adjacent to both x_0 and x_1 . By Lemma 5.9, for $i = 0, 1$, there exists a vertex $a_i \in S_G(u, 1)$ which is adjacent to both x_i and y . These vertices a_0 and a_1 are distinct since, by hypothesis, no vertex in $S_G(u, 1)$ is adjacent to both x_0 and x_1 . Let $b \in N_G(x_0) \cap S_G(u, 1) - \{a_0\}$. By Lemma 5.8, a_0 is adjacent to both b and a_1 (note that b is distinct from a_1 , otherwise b would be adjacent to both x_0 and x_1 contrary to the hypothesis). Since the cycle $C = \langle u, b, x_0, y, a_1, u \rangle$ is well-bridged and since x_0 and y are the antipodes of u in C with $d_C(u, x_0) = d_C(u, y) = 2$, the vertices b and a_1 must be adjacent.

Now consider the cycle $C' = \langle x_0, y, a_1, b, x_0 \rangle$. The vertex b is the antipode of y in C' , and x_0 and a_1 cannot be adjacent by hypothesis. Therefore, y and b must be adjacent, which proves the lemma. \square

Lemma 5.12. *Let F be a finite subset of $V(G)$. Then there exists a finite $A \subseteq S_G(u, 2)$ which contains $F \cap S_G(u, 2)$ and such that $B_G(u, 1) \cup A$ induces an isometric subgraph of G .*

Proof. Construct a sequence A_1, A_2, \dots of finite subsets of $S_G(u, 2)$ as follows. Put $A_1 := F \cap S_G(u, 2)$. Suppose that A_1, \dots, A_n have already been constructed for some

$n \geq 1$. For every i and j such that $i + j = n + 1$, and for each $(x, y) \in A_i \times A_j$ with $d_G(x, y) = 2$ and such that there exists no (x, y) -geodesic in the subgraph of G induced by $S_G(u, 1) \cup A_1 \cup \dots \cup A_n$, choose a vertex z in $S_G(u, 2)$ which is adjacent to both x and y . Such a vertex exists by the convexity of $B_G(u, 2)$. Let A_{n+1} be the set of all such vertices z .

By Lemma 5.11 and by the construction, for every $n \geq 1$, the neighborhood of each element of A_n contains the neighbors in $S_G(u, 1)$ of n vertices of A_1 . Put $A := \bigcup_{n \leq p} A_n$ where $p := |A_1|$. Therefore, we are done by Lemma 5.10 and since every pair of elements of A whose distance is 2 have a common neighbor in $S_G(u, 1) \cup A$. \square

Proof of Theorem 5.6. Assume that $V(G) = B_G(u, 2)$ for some vertex u of G such that $G[N_G(u)]$ contains no infinite simplices, and let H be a finite subgraph of G . By Lemma 5.12, there exists a finite $A \subseteq S_G(u, 2)$ such that $V(H) \cap S_G(u, 2) \subseteq A$ and $G' := G[\{u\} \cup V(H) \cup A \cup \bigcup_{a \in A} (N_G(a) \cap S_G(u, 1))]$ is an isometric subgraph of G . In particular, G' is a bridged graph which contains H as a subgraph. Moreover, G' is finite because each $N_G(a) \cap S_G(u, 1)$ is finite by Lemma 5.8 and the fact that $G[N_G(u)]$ contains no infinite simplices. \square

6. Weakly cop-win graphs

As we already said, in the finite case the constructible graphs are exactly the cop-win ones. However, in the infinite case this is no longer true, and the definition of a cop-win graph is then very restrictive. In fact, even trees may not be cop-win. In this section we introduce the following generalization of a cop-win graph. We will say that, in the pursuit game imagined by Nowakowski and Winkler [11], a graph is *weakly cop-win* if the cop wins either if he really catches the robber or if he forces him to run straight ahead, that is, move endlessly by visiting each vertex at most once, except possibly finitely many of them at the beginning of the game. Note that this new concept coincides with the original one in the finite case. Moreover trees are all weakly cop-win and, as we will see later, so are all connected bridged graphs and all Helly graphs.

Theorem 6.1. *Let G be a constructible graph admitting a constructing order \leq to which is associated a domination map Δ which is a self-contraction of G . Then G is weakly cop-win.*

Proof. Construct inductively a sequence of vertices c_0, c_1, \dots and a sequence of non-negative integers i_0, i_1, \dots such that, for $n \geq 0$, $i_n \geq i_{n+1}$ and $c_{n+1} = \Delta^{i_n}(r_n)$ where r_n is the n th position of the robber.

Let $c_0 := u$ where u is the smallest vertex of G with respect to \leq , and let i_0 be the smallest non-negative integer such that $\Delta^{i_0+1}(r_0) = u$. Put $c_1 := \Delta^{i_0}(r_0)$. Suppose that c_0, \dots, c_n and i_0, \dots, i_{n-1} have already been constructed for some $n \geq 1$; and let r_n

be the n th position of the robber. Note that if Δ is a contraction, then Δ^i is also a contraction for every non-negative integer i . We distinguish two cases.

Case 1: $r_n > r_{n-1}$.

Then, since r_n and r_{n-1} are adjacent, $\Delta^{i_{n-1}}(r_n)$ is equal or adjacent to $\Delta^{i_{n-1}}(r_{n-1})$ which is equal to c_n by the induction hypothesis. Put $i_n := i_{n-1}$ and $c_{n+1} = \Delta^{i_n}(r_n)$.

Case 2: $r_n \leq r_{n-1}$.

Then $\Delta^{i_{n-1}-1}(r_n)$ and $\Delta^{i_{n-1}-1}(r_{n-1})$ either coincide or are adjacent. Hence $c_n = \Delta^{i_{n-1}}(r_{n-1})$ and $\Delta^{i_{n-1}-1}(r_n)$ either coincide or are adjacent since c_n dominates $\Delta^{i_{n-1}-1}(r_{n-1})$ in $G[\{x \in V(G) : x \leq \Delta^{i_{n-1}-1}(r_{n-1})\}]$. Put $i_n := i_{n-1} - 1$ and $c_{n+1} = \Delta^{i_n}(r_n)$.

Note that, if $r_n \leq r_{n-1}$ for more than i_0 integers n , then there exists p such that $c_p = \Delta^0(r_{p-1}) = r_{p-1}$, which means that the robber is caught. Therefore, if the robber wants to avoid this event, we must have $r_n > r_{n-1}$ for every n greater than some non-negative integer m , which means that the robber runs straight ahead. \square

We will say that a graph G has property (P) if, for every $u, x \in V(G)$ with $d_G(u, x) = r > 1$, there exists $y \in N_G(x)$ such that $d_G(u, y) = r - 1$ and such that y is adjacent to every neighbor z of x with $d_G(u, z) \leq r$.

Lemma 6.2. *If a connected graph G has property (P) , then every BFS-order on $V(G)$ is a constructing order to which is associated a domination map which is a self-contraction of G .*

Proof. Let \leq be a BFS-order on $V(G)$ with u as the smallest element. We will construct a domination map Δ associated with \leq as follows. Put $\Delta(u) := u$. Let $x \in V(G - u)$ and $r := d_G(u, x)$. Put $\Delta(x) := u$ if $r = 1$. Suppose $r > 1$. By property (P) there exists $y \in N_G(x)$ such that $d_G(u, y) = r - 1$ and which is adjacent to every neighbor z of x with $d_G(u, z) \leq r$. Put $\Delta(x) := y$. Then $\Delta(x) < x$ and, by definition, $\Delta(x)$ dominates x in $G[\{y \in V(G) : y \leq x\}]$. Consequently Δ is a domination map associated with \leq .

It remains to show that Δ is a self-contraction of G . Let x and y be two adjacent vertices of G . W.l.o.g. we can suppose that $x < y$. Then x and $\Delta(y)$ either coincide or are adjacent. We are done if $x = \Delta(y)$ since $\Delta(x)$ is adjacent to x . If $x \neq \Delta(y)$, then $d_G(u, \Delta(y)) = d_G(u, y) - 1 \leq d_G(u, x)$ because x and y are adjacent and $x < y$. Therefore, by the definition of $\Delta(x)$ and the fact that x and $\Delta(y)$ are adjacent, $\Delta(x)$ and $\Delta(y)$ either coincide or are adjacent, which completes the proof. \square

Theorem 6.3. *Every connected graph having property (P) is weakly cop-win.*

This is an immediate consequence of Theorem 6.1 and of Lemma 6.2. This result has an interesting consequence in the particular case of Helly graphs.

Lemma 6.4. *A cop-win graph contains no isometric rays.*

Proof. Suppose that a graph G contains an isometric ray $R = \langle x_0, x_1, \dots \rangle$, and let c be the starting position of the cop. By Polat [15, Lemma 3.7], there exists $n \geq 0$ such that $x_n \in I_G(c, x_p)$ for every $p \geq n$. Therefore a winning strategy for the robber will be to choose as a starting position a vertex x_p for any $p > n$, and then move endlessly along R . \square

Theorem 6.5. *Every Helly graph is weakly cop-win. Moreover, a Helly graph is cop-win if and only if it contains no isometric rays.*

Proof. We claim that any Helly graph G has property (P). Let $u, x \in V(G)$ with $d_G(u, x) = r > 1$. The balls $B_G(u, r-1)$, $B_G(x, 1)$ and $B_G(y, 1)$ for every $y \in N_G(x) \cup B_G(u, r)$ are pairwise non-disjoint. Hence, since G is a Helly graph, their intersection is non-empty, which proves the claim.

The first part of the statement is then a consequence of Theorem 6.3. The fact that a Helly graph is cop-win if it contains no isometric rays was proved by Polat [14, Theorems 5.3 and 6.2]. The converse is a consequence of Lemma 6.4. \square

We will now consider the case of bridged graphs. We recall that a subset A of the vertex set of a graph G is said to be *bounded* (in G) if its diameter $\sup\{d_G(x, y) : x, y \in A\}$ is finite. We will say that a ray R in a graph G is *partly bounded* if some infinite subset of $V(R)$ is bounded in G .

Theorem 6.6. *Let G be a connected bridged graph. Then G is weakly cop-win. Moreover, G is cop-win if it contains no infinite simplices and if all its rays are partly bounded.*

Proof. The first part is a consequence of Corollary 4.5 and of Theorem 6.1. The second part was proved by Polat [16, Remarks 3.13]. \square

7. Open problems

7.1. Finite subgraphs of infinite bridged graphs

The generalization of Hahn et al.'s result [8] (see Section 5) gives rise to two questions:

Question 1: Is every finite subgraph of a bridged graph G of diameter n ($n \geq 1$) contained in a finite subgraph of G which is bridged and has diameter n ?

Question 2: Is every finite subgraph of a bridged graph G contained in a finite isometric subgraph of G ?

In a recent paper [17] Polat answered this question in the affirmative in the particular case of bridged graphs that contain no infinite simplices. Note that an affirmative answer

to Question 2 will give an affirmative answer to Question 1 since an isometric subgraph of a graph G obviously has a diameter which is at most equal to that of G .

7.2. Weakly cop-win graphs

Theorem 6.1 raises three questions

Question 3: Let G be a constructible graph. Does there exist a constructing order on $V(G)$ admitting an associated domination map which is a self-contraction of G ?

Question 4: Are the weakly cop-win graphs exactly the constructible ones?

Question 5: Let G be a weakly cop-win graph. Does there exist a constructing order on $V(G)$ admitting an associated domination map which is a self-contraction of G ?

Affirmative answers to both Questions 3 and 4 obviously give an affirmative answer to Question 5.

Finally, the conditions given in Theorem 6.6 for a connected bridged graph to be cop-win are sufficient but not necessary, as was shown by Polat [16, Remarks 3.13]. Furthermore, the condition of containing no isometric rays for a graph to be cop-win, necessary by Lemma 6.4, is not sufficient for bridged graphs. In fact, Hahn et al. [8] showed that there exists a connected bridged (actually chordal) graph G of diameter two which is not cop-win, but such that for every vertex u , $G - N_G(u)$ contains infinite simplices. Whence the last question:

Question 6: Which infinite connected bridged graphs are cop-win?

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